The results also indicate extraordinary sensitivity to node-point distribution, especially near the wall, and suggest that considerable improvement in accuracy, regardless of solution method, would accrue if an "optimal" distribution could be found. In this respect, the authors have developed a new method for two-point boundary value problems [8] which shows promise for marching problems as well.

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A COMMENT ON THE PERIODIC FREEZING AND MELTING OF WATER

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INTRODUCTION

Lock *et al.* [1] have presented results for the periodic freezing and melting of water in an essentially undimensional cartesian arrangement wherein the temperature at the plane $x = 0$ varied in an approximately sinusoidal manner, with the mean value being the fusion temperature. The system initially was liquid so that freezing occurred in the first half period, with melting in the second half period, and the two hour period was executed successively thereafter. Figure 1 shows by points some of the results in terms of the depth in centimeters and the time in hours, with the points being taken from the first, second and fourth freeze and the first and third melt. Lock *et al.* showed that a good degree of correspondence was achieved both by approximate analysis and numerical solution of the conduction equation for the developing phase; the temperature in the existing phase being the fusion value. The analytic solution is not easy to evaluate and the numerical solution appears to have been prodigal of computer time.

In the water–ice system, the latent heat of fusion is large enough so that when the surface temperature is not far from the saturation value then the thermal capacity effects in the solid can be ignored (low Stephan number). Then the solution [2] of the problem involves conduction effects alone and is quite simple. Conversely, tests with water cannot really verify the degree to which a theory adequately accounts for the thermal capacity effects unless the surface temperature amplitude is made very large.

It is the present purpose to show that the simple theory adequately predicts the experimental features of Lock's results to the extent that the more complicated analysis does so. Lock has, in fact, already done this in a prior reference [3]. Also, there is examined his suggestion that some of the difference between theory and experiment is ascribable to the effect of convection as that has been indicated by the melting experiments of Yen [4]. This consideration shows that while the convective effect probably did exist and is in the direction required, it is barely discernible in terms of the results of the Lock experiments. Generally, however, the effect is important and should be considered in melting problems.

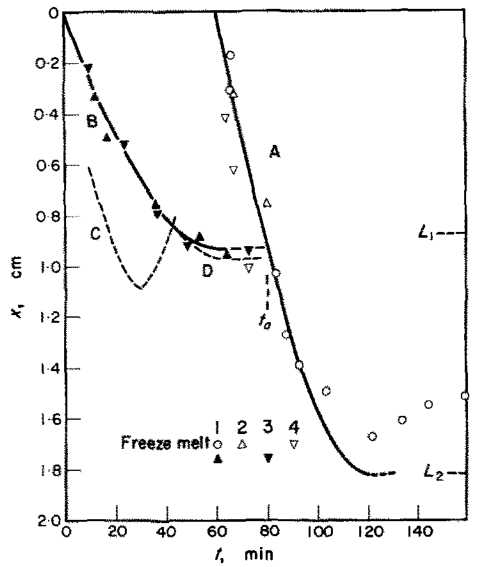

FIG. 1. Depths of freezing and melting. Curves are predictions: A, freezing; B, melting; D, melting with convection; C, critical depth for onset of convection. L_1 and L_2 are the terminal values for melting and freezing as predicted in [1].

SIMPLE THEORY

The neglect of the thermal capacity of the advancing phase and the assumption of saturation temperature for the other phase yields the simple energy balance that defines the rate of advance of the phase interface. Taking L as the heat of fusion per unit volume and k as the thermal conductivity and the fusion temperature as T_s

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{L}\frac{|T_0 - T_s|}{x}. \qquad (1)$$

Integration is simple for any kind of variation of the surface temperature, T_0, but for the periodic case only positive values of $(T_0 - T_s)$ can be considered because the advance of the developing phase can only be positive. This is true also of the analyses of [1] and because of this the equilibrium thickness of the frozen layer is analytically unavailable. For the periodic case, in which $T_0 - T_s = A \sin \omega t$, the rate of growth given by equation (1) is

$$x = \frac{2kA}{L\omega}(1 - \cos \omega t) \qquad 0 < \omega t < \pi. \qquad (2)$$

For freezing of water $L/2k = 7{\cdot}10$ h°C/cm² and for melting $L/2k = 1{\cdot}92$ h°C/cm². The experimental temperature variation was $10 \sin \pi t$, °C, with the time in hours. Curve A of Fig. 1 represents equation (2) for ice; it represents the first and ensuing freezings and begins at the half period time. Curve B represents equation (2) for water and represents the first and ensuing meltings; the first melting takes place a half period after the first freezing initiates the period scale.

Initially the region $x > 0$ is all liquid at the freezing temperature and, with the surface temperature diminishing initially, freezing takes place according to Curve A until after a half period the surface temperature returns to the freezing value. The predicted thickness there is greater than the measured value shown by the circles. There is no explanation for the 7 per cent error in predicted thickness at this point. After this half period, when melting has already begun at $x = 0$, there is a reduction in depth of the lower ice interface. This is also unexplained.

The melting that begins when the surface temperature rises above the freezing value is specified by Curve B, which terminates at the half period, when the surface temperature returns to the freezing value. Neglecting thermal capacity effects, this position of the melting front is then stable at this value; a dashed extension of Curve B indicates this; this depth defines the "active zone." For the ensuing freezing cycle Curve A really terminates at this depth, at time t_a. For time $t_a < t < 120$ min, the prediction according to equation (1) ought to be continued, with the depth of the ice layer from the previous period as the initial condition. This leads to a continuous thickening of the lower boundary of the ice, at a rate decreasing as the thickness increases. This aspect does not conform to the implied thickness of this layer, which [1] indicates to have remained at the value shown by the last circle on Fig. 1.

Figure 1 demonstrates that the simple theory explains the observed performance in the active zone as adequately as does the more complicated theory of [1]. It was indicated there that the slightly greater depth of the active zone that what is predicted might be due to the convective effects in the melting process that have been demonstrated by Yen [4]. Even though such effects are indeed small for Lock's experimental conditions, it is appropriate and interesting to analyze further Yen's results to obtain a prediction in which convective effects are included.

CONVECTIVE EFFECTS IN THE LIQUID LAYER

With the distance x measured vertically downward, as in Lock's experiments, the layer of water just above the ice, in which the temperatures are below 8°C, contains fluid in which the density increases with height. Instability can occur and a convection pattern can be established in this region, and for a fluid with a uniform coefficient of expansion, β, the onset of such motions is expected at a Rayleigh number, $(g\beta \Delta T \delta^3/\nu\alpha)$ of about 1200. With water, β varies greatly in the temperature range that is involved and the simplest hypothesis is to take $\beta \Delta T$ as

$$\int_0^T \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}T}\,\mathrm{d}T\,;$$

where v is the specific volume of the water. Then, retaining the critical Rayleigh number at 1200, δ can be evaluated for temperatures in the range considered. This is done for $0 < T < 6°C$, for at 6°C $\beta\Delta T$ attains its maximum value; it decreases for higher temperatures. If the temperature at $x = 0$ is 6°C or less, then unstability is expected if $x > \delta$. If the surface temperature exceeds 6°C then instability may develop in the lower region of the liquid layer where $T < 6°C$.

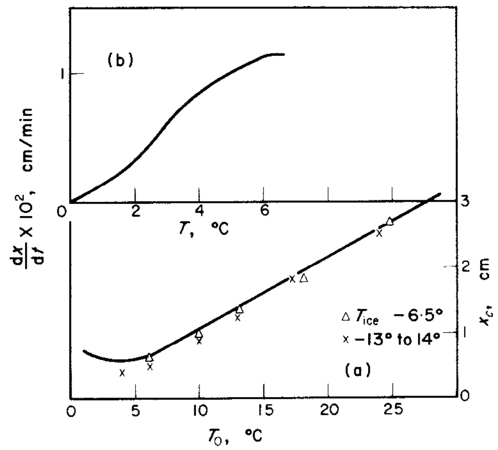

FIG. 2. Critical layer thickness and convective growth for water. Part (a) shows the critical thickness; points are the estimates of [4]. Part (b) illustrates equation (4).

Then $\delta = 6/T_0\, x$, where T_0 is in °C. In this way critical depths x_c, can be evaluated in terms of T_0; they are shown by the curve on Fig. 2. Yen measured depths of melt at which departures from equation (1) occurred in a system in which T_0 was constant and the points on Fig. 2 are some of these values. They support the prediction. Yen also measured the times at which these depths occurred and those times do not conform to those predicted by equation (1), always being too large. There is an implication that in those transient experiments the surface temperature was not truly a step in time but actually rose a bit more slowly. This discrepancy, however, does not invalidate the conclusions to be drawn from Fig. 2, which considers conditions only at the time at which the liquid layer thickness x_{cr} existed.

The dashed Curve C on Fig. 1 gives these critical depths for Lock's experiments as obtained from the curve of Fig. 2. Instability is predicted at 43 min after the beginning of melting.

Once the convective layer exists, the rate of advance of the melt is no longer given by equation (1), but it is now controlled by the heat transfer coefficient for the convective layer and the temperature difference that exists across it.

$$\frac{L}{k}\frac{dx}{dt} = \frac{h}{k}T \qquad 0 < T < 6°C. \tag{3}$$

When $T > 6$ there is a layer of stable water above the convective region and the remainder of the temperature rise occurs across it by conduction.

Experiments on horizontal layers of expansive fluids heated from below reveal that $h\delta/k \sim [g\beta\Delta T\delta^3/v\alpha]^n$, where $n = 0{\cdot}25$ for $Ra < 10^5$ and $n = 0{\cdot}30$ for $Ra > 10^5$. Since equation (3) usually must be integrated numerically, the use of the correct formulations for h/k is involved but not difficult. Here, however, there is used the simplification proposed by Yen, wherein his results appear to agree adequately with $n = \frac{1}{3}$, so that h/k becomes independent of δ. He specifies the proportionality factor of 0·08, though here, to obtain a better approximation in the low Rayleigh number range, 0·10 is used. Equation (3) then becomes

$$\frac{dx}{dt} = \frac{k}{L}\left[0{\cdot}10\left(\frac{\int_0 \frac{1}{v}\frac{dv}{dT}\,dT}{v\alpha}\right)^{\frac{1}{3}}\right]T \qquad 0 < T < 6. \tag{4}$$

Figure 2 shows dx/dt so evaluated. For surface temperatures greater than 6°C, dx/dt is the same as it is for 6°C. For Lock's experimental conditions, the numerical integration of equation (4) onward from the time at which the critical depth is assumed gives Curve D of Fig. 1. The melting depth is only slightly increased by the convection which is indicated to exist for the period $43 < 4 < 60$ min, while the surface temperature is in the range $7{\cdot}8 > T_0 > 0$. This agrees with the trend of Lock's data.

CONCLUSION

The simple theory of freezing or melting is adequate to represent the periodic freezing and melting as measured by Lock. The effect of convection is small in that experiment, but its onset appears to conform to expectation in Yen's experiments and those support the method used to include convective effects in the prediction for Lock's experiments.

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